Research Statement

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1 Introduction

A significant amount of the work on discrete groups in a geometric setting stems from attempts to discretize the behavior of continuous groups. Hyperbolic groups, discussed below, were first studied in the context of groups which act by isometries on hyperbolic n–space, and so importantly are subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. This relationship is often made precise through the use of *lattices*, discrete subgroups of an isometry group associated to a Riemannian manifold (or other kind of continuous group, occasionally), which act with finite covolume on the manifold. I will first outline my work related to these discrete groups, then discuss how it has been motivated by certain Lie groups and isometries thereof. Then we will examine a generalization of those metric groups, using a left-invariant metric that I constructed. The penultimate section includes some unposted work that is very much in progress, while the last section is focused specifically on research to be conducted at the University of Jyväskylä.

2 Background

Discrete groups that arise from a topological setting are often studied using geometric group theory. In this discipline, group theory is done coarsely, using a variety of geometric and topological tools. Our (discrete) groups are categorized into equivalence classes such that, for example, all finite groups are in the same class as the trivial group. These equivalence classes come from the metric on a Cayley graph for the group. We say *a* Cayley graph, because for any finitely generated group, any finite generating set will give graphs that we consider equivalent. This equivalence is called quasi-isometry on metric spaces, which allows us to study the large scale geometry of a space. This classification is able to pick up information like ends, boundaries, and coarse versions of curvature, while throwing out a lot of information we aren't interested in keeping track of, such as exact distances. The correspondence is made exact by a result of Svarc and Milnor, printed succinctly in [4], which states that two spaces are quasi-isometric if they both admit actions by the same group which are geometric - a shortcut for properly, cocompactly, and by isometries.

One large-scale geometric invariant that is important to us is $\delta-$hyperbolicity, or simply hyperbolicity, which states that geodesic triangles are coarsely 'thinner' than in Euclidean space. A number of strong statements can be made about the class of δ–hyperbolic spaces, such as exponential divergence of geodesics and linear isoperimetric inequality. We will apply the label of $\delta-$hyperbolicity to a group if it admits a geometric action on such a space. However, this condition is restrictive, and some of the most infamous groups in geometric group theory fail to be hyperbolic, including those non-cocompact lattices in negatively curved Riemannian manifolds. An idea came down to us from Gromov, popularised by Bowditch, that we could expand the class of groups we make statements about by generalizing the action we allow. While we lose some properties, we retain the ability to talk about a well-defined homeomorphism type boundary, often called the Bowditch boundary, of a *relatively hyperbolic* group. A natural metric structure on this space is less clear, and this will be the inspiration for some of the work below.

The boundary of a hyperbolic space is a compactification of the space which is well-defined up to quasi-isometry of the space. The action of a group by isometries always extends to an action on the boundary, which can be thought about as equivalence classes of geodesic rays emanating from a base-point. These topological spaces actually carry with them a surprisingly large amount of algebraic data about the group. As such, one favored past-time of geometric group theorists is finding and analyzing these compacta for particular groups. For example, when they have local cut points can tell us that our group will split as an amalgamated free product. We will see the importance of the boundary as well in the continuous group setting, as Heintze groups (to be defined later, using the term Heintze spaces), have boundaries with a distinguished point that is preserved under all isometries.

Because this boundary comes from the metric space, we find we have the ability to assign a boundary to a group by assigning it a hyperbolic space on which it acts. However, we should be careful about what the correct space to consider is. In the case of a geometric or freely transitive action, it is clear. As noted above, we get a topological but not metric answer for relatively hyperbolic groups. Even less is known in the case of further generalizations, such as acylindrical hyperbolicity - see [6], [8].

3 Metric Boundaries and Relatively Hyperbolic Groups

First we will consider the class of relatively hyperbolic groups mentioned above. For an intuition of what this class encompasses, one may consider finite-volume lattices of hyperbolic n–space which are not cocompact (with compact quotient), such as fundamental groups of hyperbolic knot complements. In general, such a group will admit a non-cocompact, but proper and by isometries, action on a hyperbolic space which admits an equivariant excision of 'cusps'. The Bowditch boundary, aptly named for the result in [2], is an invariant of the group when one specifies a finite family of subgroups, called 'peripheral subgroups', conjugates of which will correspond the excised cusps. This pair is often denoted $(\Gamma, \mathbb{P})$, and the action 'cusp-uniform'. I show this boundary homeomorphism is not guaranteed to be induced by an appropriate quasi-isometry of the spaces.

Theorem 3.1. *[6] Let $(\Gamma, \mathbb{P})$ be a relatively hyperbolic pair where at least one peripheral subgroup is infinite. Then there exist hyperbolic spaces X_1, X_2 that admit cusp-uniform actions by $(\Gamma, \mathbb{P})$ which are not equivariantly quasi-isometric.*

This theorem shows that when you relax the conditions on hyperbolicity of your group, you lose some rigidity of the relevant spaces. Any two spaces acted on geometrically by a hyperbolic group are quasi-isometric, and any two equivalent relatively hyperbolic structures on a group have homeomorphic Bowditch boundary even if they fail to be quasi-isometric.

The proof of the preceding theorem exploits a way of breaking quasi-isometric maps which distorts a space far from the natural example of a negatively curved Riemannian manifold such as $\mathbb{H}^n$. We may hope to recover some rigidity if one insists on conditions that will mimic the geometry we find in rank one symmetric spaces. In fact, my collaborator Chris Hruska and I show this is true.

Theorem 3.2 (Healy-Hruska, 2019)**.** *Any two hyperbolic spaces which are visual and have constant horospherical distortion that admit cusp-uniform actions by a relatively hyperbolic pair $(\Gamma, \mathbb{P})$ are quasi-isometric and have equivariantly quasi-symmetric boundary.*

The second condition above insists that we do not warp the space too much beyond the natural negative curvature found in a negatively curved symmetric space. The first condition insists that the space does not admit larger and larger bounded subsets as one moves into a cusp. Again this is naturally satisfied in the motivating examples.

We see there is an additional conclusion from the statement, one about the boundary. Going back to the homeomorphism type of the boundary found in [2], we see that this rigidity was not induced by a quasi-isometry - it was a consequence of the group. An actual quasi-isometry between spaces will actually give us more information than just the shape of the boundary. In fact it will induce a *quasi-symmetry*, which is a homeomorphism that carries some information about the natural metric on said boundary. This metric is often aptly called the visual metric.

From this well-defined quasi-isometry class of hyperbolic space and quasi-symmetry type of the boundary, we can go on to extract a lot of useful algebraic information from the pair. As an example, the following is a corollary of Theorem 3.2. We use 'model space' as shorthand to denote a space admitting a cusp-uniform action that is both visual and has constant horospherical distortion.

Corollary 3.3. *A finitely generated group is virtually free if and only it has some family of subgroups such that the associated model space is quasi-isometric to* $\mathbb{H}^2$.

I expect many such statements will be able to be proved from the machinery developed to make the statement above. In particular, I would like to know what other classes of groups are distinguished by their model space. We might ask, for example, how we can recognize groups virtually isomorphic to more generic free products geometrically, given that these are hyperbolic relative to the factors.

4 Carnot Groups and Heintze Spaces

As indicated, the motivating examples above came from non-cocompact lattices acting on hyperbolic space. Indeed, real, complex, and quaternionic hyperbolic spaces (and the octave hyperbolic plane), are all 'hyperbolic' in the large-scale geometric sense, and serve as the quintessential examples of negatively curved Riemannian manifolds. To expand on results of groups which act on such spaces, we take a closer look at their structure. Indeed, these rank one symmetric spaces of noncompact type are all examples of Heintze spaces.

Definition 4.1. A *Heintze space* is a connected Riemannian manifold of strictly negative sectional curvature which is homogeneous.

Homogeneity above actually secretly encodes the information we are concerned with - a space is homogeneous if its isometry group acts transitively on the space. In essence, the space looks the same at every point. The reason for the name Heintze space is because of a fundamental result from [9], which states that any such space is actually isometric to a Lie group of the form $N \rtimes \mathbb{R}$ with a certain left-invariant metric and N a nilpotent Lie group. In fact, there is even more structure we can ascribe to this decomposition, such as the nature of the $\mathbb{R}$ action on N, which is *contracting* in the appropriate sense (see [10]). In special cases, a Heintze space may also adhere to the below condition on the nilpotent piece.

Definition 4.2. A *Carnot grading* on a nilpotent Lie algebra $\mathfrak{n}$ is a decomposition

$$\mathfrak{n} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \ldots \oplus \mathcal{V}_s$$

with the following properties:

$$[\mathcal{V}_1, \mathcal{V}_j] = \mathcal{V}_{j+1} \qquad [\mathcal{V}_1, \mathcal{V}_s] = 0$$

A nilpotent Lie group whose algebra admits a Carnot grading is called a *Carnot group*.

In light of the above definitions, we call a Heintze space of *Carnot-type* when the nilpotent piece of its semidirect product decomposition admits a Carnot grading. It is not difficult to see that the rank one symmetric spaces of noncompact type are of Carnot-type.

Heintze's seminal work in [9] also guarantees the other direction is valid too - any Lie group of the form $N \rtimes \mathbb{R}$ which satisfies the contracting automorphism condition will admit some left-invariant metric of negative curvature. Unlike the symmetric spaces, however, there is not immediately a natural candidate metric of negative curvature to choose. My main result in [7] is to motivate and prove the existence of a metric which generalizes the symmetric space metrics.

To give context, we recall a fact which is well known about symmetric metrics on hyperbolic spaces. The following may be obtained by a careful interpretation of [5], but is proven as stated in [7].

Proposition 4.3. *[5] Consider the manifold $\mathbb{KH}^n$ viewed as an upper half-space with derived subalgebra $\mathfrak{heis} = \mathbb{K}^n \oplus \mathrm{Im}(K)$, normalized so that the supremum of its sectional curvature is -1. This space has the following curvature properties ($V, W \in T_p\mathbb{KH}^n$, V vertical):*

$$K(V, W) = -1 \text{ if } W \in (dL_p)_e \mathbb{K}^n$$
$$K(V, W) = -4 \text{ if } W \in (dL_p)_e \mathrm{Im}(K)$$

If we recognize the decomposition of the Heisenberg subalgebra above as its Carnot grading, we see the pattern that the curvature determined by the tangent plane when considering a vertical vector (those orthogonal to the nilpotent subalgbera) is $-i^2$ when the second basis element is in $\mathcal{V}_i$ for the grading $\mathfrak{n} = \oplus_i \mathcal{V}_i$. This gives us the means to define a generalization of this metric for arbitrary Heintze spaces of Carnot-type. In the following theorem, a layered basis is a basis for the Lie algebra which respects the Carnot grading.

Theorem 4.4. *[7] For any Heintze space of Carnot-type M with stratification $\oplus_i \mathcal{V}_i$, there exists a Riemannian metric on M such that at any point $p \in M$, if X, Y are layered basis tangent vectors in T_pM, then the sectional curvature $K(X, Y)$ satisfies the following conditions:*

- *$K(A, Y) = -i^2$ for vertical planes such that $Y \in \mathcal{V}_i$, and*
- *$-1 \leqslant K(X, Y) \leqslant -s^2$ where s is the step nilpotency of the base group.*

As an added benefit to the above construction, we see that not only does a metric exist on these Lie groups which generalizes the curvature pattern of symmetric spaces, but also one which generalizes the pinched-curvature property as well (I prove this stronger claim later in [7]). In particular, we see that a Heintze group of Carnot-type whose Carnot group is nilpotent of step s admits a metric which has $\frac{1}{s^2}$–pinched curvature (this value represents the ratio of the largest to the smallest sectional curvature values). In fact, in a special case, I find this value is optimal – observe this is the case for symmetric spaces.

Theorem 4.5. *[7] Let M be a Heintze space of Carnot-type whose Carnot subgroup admits a lattice. Then M is not C–pinched for any $C > \frac{1}{s^2}$, where s is the step nilpotency of the derived subgroup.*

Though it was known already, an easy corollary of the existence of this metric is that such spaces do not admit lattices. I derive several other interesting results regarding these metrics in [7].

5 Beyond Heintze Spaces

The work in this section represents an ongoing project, which is joint with Mark Pengitore of Ohio State University.

One may ask, in the above setup of Heintze spaces of Carnot-type, what the importance of the righthand group having rank 1 is. Of course, it is a consequence of Heintze's theorem that it must, if it admits a metric of negative sectional curvature. Therefore, if we want to consider higher-rank spaces, we must expect a weaker geometric conclusion. Define an *orthant group* to be a Lie group of the form $N \rtimes \mathbb{R}^k$ where N is a Carnot group and the semi-direct product is contracting when restricted to each factor of R^k. The following is a result which is not yet posted.

Theorem 5.1 (Healy-Pengitore, 2020, to be completed)**.** *Orthant groups admit* CAT(0) *metrics.*

As a consequence, we may conclude strong properties about the discrete groups which act on them.

Corollary 5.2. *Orthant groups which are not symmetric do not admit lattices.*

The proof of the corollary is obtained by applying strong restrictions to groups which can act properly and cocompactly on CAT(0) spaces, such as are present in [3].

6 Work to be Done at University of Jyväskylä

I plan to continue my work into metric groups, especially Carnot groups and related objects. Both the Heintze spaces of Carnot-type and the above orthant groups are examples of groups which are semidirect products that have normal Carnot subgroups. We may therefore ask the following question.

Question 6.1. *What are the metric properties of general extensions of Carnot groups? What can the Riemannian geometry tell us about the existence of lattices in general?*

I would also like to answer a natural question that arises from my work in [7].

Conjecture 6.2. *The existence of a lattice in the Carnot group can be dropped when observing the pinched-curvature tightness from [7].*

I strongly suspect the above conjecture is true. However, the proof technique will not adapt from the lattice case, as the proof relies on the main theorem from [1], which assumes the existence of a proper, cocompact action. Therefore I would be eager to collaborate with a group that has significant expertise in Carnot groups.

It is observed in [10] that not all nilpotent Lie groups admit Carnot gradings. Therefore, it is of interest to know how much of the existing machinery in this case goes through for generic Heintze spaces.

Question 6.3. *What are some pinched-curvature statements that can be made about Heintze spaces whose derived subgroup is not a Carnot group? How does this depend on the step of nilpotency of the derived subgroup?*

I expect to get some use out of the statement from [12] which says that Carnot groups are generic in the sense that the asymptotic cones of all nilpotent Lie groups are Carnot groups. In this way, it is possible we find a supremum constant for pinched-curvature without the existence of a metric which actually realizes it, which would be a departure from Carnot-type spaces.

The following discussion results from a discussion with Gabriel Pallier of GeoMeG. He observed that pinched-curvature bounds on Heintze spaces (not necessarily of Carnot-type) can also be obtained using Pansu's conformal dimension. The following are a few guiding questions which I would like to spend more time investigating with Pallier – see also his note at [11].

Question 6.4 (Healy-Pallier)**.** *How can Pansu's conformal dimension be used to obtain restrictions on left-invariant metrics for generic Heintze spaces? While not tight in the Carnot-type case, can such pinched-curvature bounds be more informational in the general setting? Can a version of this machinery be adapted to more general extensions of Carnot groups or nilpotent Lie groups?*

I believe that the metric geometry focus at the University of Jyväskylä and the GeoMeG project is an ideal setting to tackle some of these questions as well as produce additional questions of interest surrounding Carnot groups and sub-Riemannian geometry.

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