Rigidity Properties for Hyperbolic Generalizations

Burns Healy

University of Wisconsin - Milwaukee

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Theorem (Gromov)

Boundaries of hyperbolic groups are well defined.

Theorem (Svarc-Milnor)

If a group acts on two spaces geometrically, those spaces are quasi-isometric.

Motivating question: How much of this topological and metric rigidity do we get to keep as we loosen the conditions on group actions on hyperbolic spaces?

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Spoiler: Not much.

Definition

An metric space action $G \curvearrowright X$ is called **acylindrical** if for every $\epsilon > 0$ there exist $R(\epsilon), N(\epsilon) > 0$ such that for any two points x, y such that $d(x, y) \geq R$, the set

$$\{g \in G | d(x, g.x) \leq \epsilon, d(y, g.y) \leq \epsilon\}$$

has cardinality less than N.

Definition

A group is called **acylindrically hyperbolic** if it acts nonelementarily and acylindrically on a hyperbolic space.

We see hyperbolic and relatively hyperbolic groups meet this criterion, so we may think of acylindrical hyperbolicity as a further generalization of relative hyperbolicity.

Acylindrically hyperbolic groups form a respectably wide class, including:

- ‘Most’ Mapping Class Groups
- Braid Groups
- $\mathrm{Out}(F_n)$ for $n \geq 2$
- Indecomposable RAAGs and RACGs
- 1 Relator, $\geq$3 Generator Groups
- ‘Most’ 3-Manifold Groups

Definition

For an acylindrically hyperbolic group G, we call an element $g \in G$ a **generalized loxodromic** if it acts as a loxodromic for some acylindrical action on a hyperbolic space.

It is possible for a group element to act loxodromically for some acylindrical action on a hyperbolic space, but elliptically for another.

What do these elements look like?

- Pseudo-Anosovs in MCGs
- Fully irreducible (iwip) elements of Out(F_n)
- Rank one elements in CAT(0) groups
- Loxodromics in Relatively Hyperbolic Groups
- Generally ‘negatively curved directions’

Definition

A *universal* action $G \circlearrowright X$ is a group action which is acylindrical, such that the space X is hyperbolic and every generalized loxodromic acts as a loxodromic.

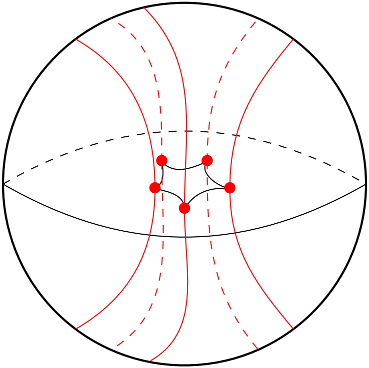
Theorem (Abbot, '17)

These actions are not guaranteed to exist.

Theorem (H.)

There exists a (finitely presented) non-elementary group G which is acylindrically hyperbolic and a hyperbolic space X such that G admits two universal, acylindrical actions, $G \circlearrowright_1 X$ and $G \circlearrowright_2 X$ such that

$$\Lambda_1(X) \ncong \Lambda_2(X).$$



Action 1 : $PSL(2,\mathbb{R}) \hookrightarrow PSL(2,\mathbb{C})$

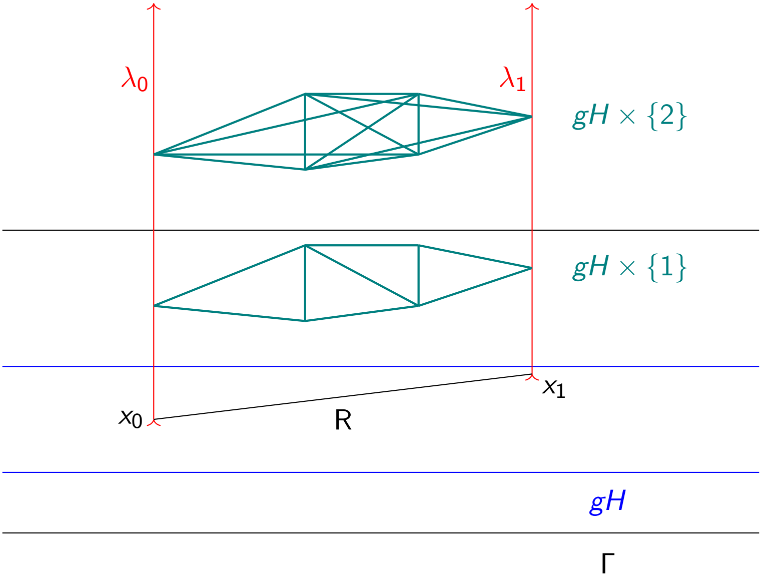
$$\begin{array}{ccc} \Sigma_2 & \longrightarrow & M_\phi \\ & & \downarrow \\ & & S^1 \end{array}$$

$$\phi \in \mathrm{MCG}(\Sigma_2), \phi \text{ pA}$$

Action 2: Hyperbolic 3-manifold

Theorem (H.)

Let G be a relatively hyperbolic group. There exist spaces X_1, X_2 that admit geometrically finite actions by G which are not (equivariantly) quasi-isometric.



Breaking Quasi-isometry in the Horoballs

Group Action Rigidity

Group Property	Action Type on Hyperbolic X	Limit Set	QI type
Hyperbolic	Geometric	Yes	Yes
Relatively Hyperbolic	Geometrically Finite	Yes	No (H.)
Acylindrically Hyperbolic	Universal	No (H.)	No

Thank you for listening!

Questions?